Skeletons for transitive fibered maps

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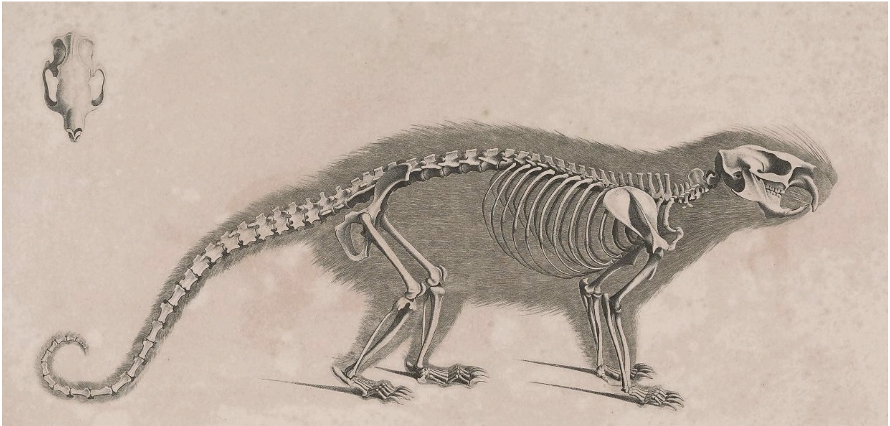
joint work with L. J. Díaz and M. Rams



Brazilian porcupine (*Coendou prehensilis*),
[Pander, Christian Heinrich, 1794-1865 - Die vergleichende Osteologie, Public Domain, https://commons.wikimedia.org/w/index.php?curid=24381421

Axioms: Some notation

Consider a one step skew-product $F\colon \Sigma_k \times \mathbb{S}^1 \to \Sigma_k \times \mathbb{S}^1$

$$F(\xi, x) = \big(\sigma(\xi), f_{\xi_0}(x)\big).$$

Consider the associated IFS $\{f_i\}_{i=0}^{k-1}$.

Some notation: Given *finite* sequences $(\xi_0 \dots \xi_n)$ and $(\xi_{-m} \dots \xi_{-1})$, let

$$f_{[\xi_0 \dots \xi_n]} \stackrel{\text{def}}{=} f_{\xi_n} \circ \cdots \circ f_{\xi_1} \circ f_{\xi_0}$$

$$f_{[\xi_{-m} \dots \xi_{-1}.]} \stackrel{\text{def}}{=} (f_{\xi_{-1}} \circ \dots \circ f_{\xi_{-m}})^{-1} = (f_{[\xi_{-m} \dots \xi_{-1}]})^{-1}$$

Given $A \subset \mathbb{S}^1$, define its *forward* and *backward orbit*, respectively, by

$$\mathcal{O}^+(A) \stackrel{\text{def}}{=} \bigcup_{n \geq 0} \bigcup_{(\beta_0 \dots \beta_{n-1})} f_{[\beta_0 \dots \beta_{n-1}]}(A)$$

$$\mathcal{O}^-(A) \stackrel{\text{def}}{=} \bigcup_{m \geq 1} \bigcup_{(\theta_{-m} \dots \theta_{-1})} f_{[\theta_{-m} \dots \theta_{-1}.]}(A)$$

Axioms: Let $J \subset \mathbb{S}^1$ be a closed *blending interval*.

T

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CEC+(J) (Controlled Expanding forward Covering).
$\exists K_1, \dots, K_5$: for every interval $H \subset \mathbb{S}^1$ intersecting J with $|H| < K_1$

- $\exists(\eta_0 \dots \eta_{\ell-1})$, $\ell \le K_2 \,|\log |H|| + K_3$, such that
$$f_{[\eta_0 \dots \eta_{\ell-1}]}(H) \supset B(J, K_4),$$
- $\forall x \in H$
$$\log \left|(f_{[\eta_0 \dots \eta_{\ell-1}]})'(x)\right| \ge \ell K_5, \quad K_5 > 1.$$

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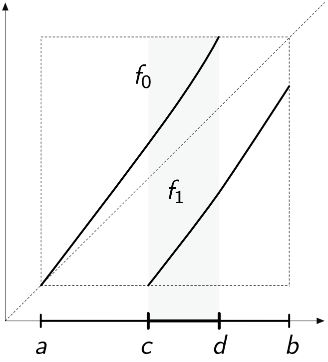
Examples. System that satisfies Axioms T, CEC±, Acc±

One-dimensional blenders

Motivated by: [Bonatti, Díaz '96], [Bonatti, Díaz, Ures '02]



IFS $\{f_i\}_{i=0}^{k-1}$, $k \geq 2$, has *expanding blender* if: there are $[c,d] \subset [a,b] \subset \mathbb{S}^1$ so that

- (expansion) $f_0'(x) \geq \beta > 1$ $\forall x \in [a,b]$
- (boundary condition) $f_0(a) = f_1(c) = a$
- (covering and invariance) $f_0([a,d]) = [a,b]$ and $f_1([c,b]) \subset [a,b]$

It has a *contracting blender* if $\{f_i^{-1}\}_i$ does.

Suppose that $\forall x \in \mathbb{S}^1$ by some forward iteration maps inside an expanding blender (a,b) and by some backward iteration meets a contracting blender.

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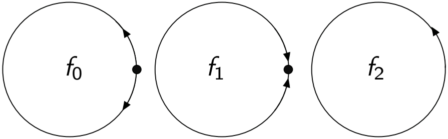
Contraction-expansion-rotation examples

Motivated by: [Gorodetskii, Il'yashenko, Kleptsyn, Nal'skii '05]



Consider IFS $\{f_i\}_{i=0}^{k-1}$, $k \geq 3$, so that

- f_0 has a repelling fixed point,
- f_1 has an attracting fixed point,
- f_2 is an irrational rotation.

Main results

Theorem (Approximating non-hyperbolic measure by hyperbolic ones)

Let $\mu \in \mathcal{M}_{\mathrm{erg}}$ *with* $\chi(\mu) = 0$ *and* $h = h(\mu) > 0$.

Then $\forall \gamma, \delta, \lambda > 0$ *there exists compact* F*-invariant transitive hyperbolic* Γ^+

$$h_{\mathrm{top}}(\Gamma^+) \geq h(\mu) - \gamma$$

and for every $\nu \in \mathcal{M}_{\mathrm{erg}}(\Gamma^+)$

$$d_{w*}(\nu, \mu) < \delta \quad \text{and} \quad \chi(\nu) \in (0, \lambda).$$

Analogously with hyperbolic Γ^- with $\chi(\nu) \in (-\lambda, 0)$ for $\nu \in \mathcal{M}_{\mathrm{erg}}(\Gamma^-)$.

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Theorem (Restricted variational principle for entropy)

$$h_{\mathrm{top}}(F) = \sup_{\mu \in \mathcal{M}_{\mathrm{erg},<0}} h(\mu) = \sup_{\mu \in \mathcal{M}_{\mathrm{erg},>0}} h(\mu) \geq \sup_{\mu \in \mathcal{M}_{\mathrm{erg},=0}} h(\mu).$$

Main results

Theorem ("Perturbing" hyperbolic measure "toward the other side")

Let $\mu \in \mathcal{M}_{\rm erg}$ *with* $\alpha = \chi(\mu) < 0$ *and* $h = h(\mu) > 0$.

Then $\forall \gamma, \delta > 0$, $\forall \beta > 0$ *exists compact* F*-invariant transitive hyperbolic* Γ

$$h_{\rm top}(\Gamma) \geq \frac{h}{1 + K_2(\beta + |\alpha|)} - \gamma$$

and for every $\nu \in \mathcal{M}_{\rm erg}(\Gamma)$

$$\frac{\beta}{1 + K_2(\beta + |\alpha|)} - \delta < \chi(\nu) < \frac{\beta}{1 + \frac{1}{\log\|F\|(\beta+|\alpha|)}} + \delta,$$

$$d_{w*}(\nu, \mu) < 1 - \frac{1}{1 + K_2(\beta + |\alpha|)} + \delta$$

Here

$$K_2 \stackrel{\rm def}{=} \inf\{K_2(J) \colon J \text{ is blending interval}\}, \quad \|F\| \stackrel{\rm def}{=} \max_{i,x}\Big\{|f_i'(x)|, |(f_i^{-1})'(x)|\Big\}.$$

Ingredients: Skeletons

F has the skeleton property relative to $J \subset \mathbb{S}^1$, $h \geq 0$, $\alpha \geq 0$ if:

There exist *connecting times* $m_{\rm b}, m_{\rm f} \in \mathbb{N}$:

$\forall \varepsilon_H \in (0,h)$ $\forall \varepsilon_E > 0$ $\exists n_0 \geq 1$ so that $\forall m \geq n_0$ there exists a finite set $\mathfrak{X} = \mathfrak{X}(h, \alpha, \varepsilon_H, \varepsilon_E, m) = \{X_i\}$ of points $X_i = (\xi^i, x_i)$:

(i) $\operatorname{card} \mathfrak{X} \asymp e^{m(h \pm \varepsilon_H)}$,

(ii) the sequences $(\xi^i_0 \ldots \xi^i_{m-1})$ are all different,

(iii) $\frac{1}{n} \log |(f_{[\xi^i_0 \ldots \xi^i_{n-1}]})'(x_i)| \asymp \alpha \pm \varepsilon_E$ $\forall n = 0, \ldots, m$.

$\exists$ sequences $(\theta^i_1 \ldots \theta^i_{r_i})$, $r_i \leq m_{\rm f}$, $(\beta^i_1 \ldots \beta^i_{s_i})$, $s_i \leq m_{\rm b}$, points $x'_i \in J$:

(iv) $f_{[\theta^i_1 \ldots \theta^i_{r_i}]}(x'_i) = x_i$,

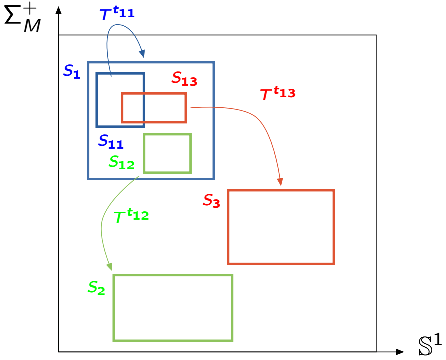
(v) $f_{[\xi^i_0 \ldots \xi^i_{m-1} \beta^i_1 \ldots \beta^i_{s_i}]}(x_i) \in J$.

Ingredients: Multi-variable-time horseshoes

Let $T\colon X \to X$ be a local homeomorphism of a compact metric space.

$\{S_i\}_{i=1}^M$ disjoint compact, $t_{ij} \in \{t_{\min}, \ldots, t_{\max}\}$ *transition times*:

$$T^{t_{ij}}(S_i) \supset S_j, \quad T^{t_{ij}}|_{S_i \cap T^{-t_{ij}}(S_j)} \text{ injective.}$$



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Let $t = t(i)$ for which $\#\{j\colon t_{ij} = t\}$ is maximal and let $A = (a_{ij})_{i,j=1}^M$

$$a_{ij} \stackrel{\text{def}}{=} 1 \text{ if } t_{ij} = t(i) \text{ and } a_{ij} \stackrel{\text{def}}{=} 0 \text{ otherwise.}$$

$\Longrightarrow$ S_{ij} and $S_{i\ell}$ are disjoint if ij and $i\ell$ are A-admissible and $j \neq \ell$.

We call $T\colon \Gamma \to \Gamma$ a multi-variable-time horseshoe, where

$$\Gamma \stackrel{\text{def}}{=} \bigcup_{k=0}^{t_{\max}-1} T^k(\Gamma'), \quad \Gamma' \stackrel{\text{def}}{=} \bigcap_{n \geq 1} \bigcup_{[c_0 \dots c_{n-1}]\, A-\text{admissible}} S_{c_0 \dots c_{n-1}}.$$

Then

$$h_{\text{top}}(T, \Gamma) \geq \frac{\log M - \log(t_{\max} - t_{\min} + 1)}{t_{\max}}.$$